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Simplification of the Constraints in the Dirac Formalism for General Relativity

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Abstract

In any classical theory in canonical form, the Poisson bracket relations between the constraints are preserved under canonical transformations. We show that in the Dirac formalism for general relativity this condition places certain limits on the degree to which one can simplify the form of the constraints. It implies, for instance, that the constraints cannot all be written as canonical momenta. Furthermore, it is not even possible to reduce them all to purely algebraic functions of the momenta by means of a canonical transformation which preserves the original configuration space subspace of phase space.

1. *Introduction*

The constraint equations which appear in the Hamiltonian formalism for general relativity have been the subject of considerable study because of their important role in the quantization program for the gravitational field. In any classical theory the presence of constraints implies that the canonical variables in terms of which the dynamical equations are written have two important defects. First of all, they are dependent, in the sense that a given set of data corresponds to a real physical field only if it satisfies the constraint equations; and secondly, they are redundant, because distinct sets of data satisfying the constraint relations represent the same real physical fields if they are related by symmetry transformations, i.e., transformations generated by the constraints. This state of affairs complicates the classical initial value problem and, in the case of general relativity, leads to serious difficulties when the standard quantization procedure is attempted (Anderson, 1963, 1964).

In 1958 Dirac demonstrated that it is possible to simplify the canonical formalism for general relativity by means of a well-chosen canonical transformation (Dirac, 1958). In terms of Dirac's canonical variables $g_{\mu\nu}$ and $p^{\mu\nu}$, the constraints may be written†

$$p^{0\mu} = 0 \quad \text{(PRIMARY CONSTRAINTS)} \tag{1.1}$$

$$\left.\begin{aligned} \mathscr{H}_s &\equiv p^{ab} g_{ab,s} - 2(g_{as}p^{ab})_{,b} \equiv -2g_{as}p^{ab}|_b \approx 0 \quad &(1.2a)\\ \mathscr{H}_L &\equiv K^{-1}(g_{ra}g_{sb} - \tfrac{1}{2}g_{ab}g_{rs})p^{ab}p^{rs} - K{}^{(3)}R \approx 0 \quad &(1.2b)\end{aligned}\right\} \text{(SECONDARY CONSTRAINTS)}$$

Here ${}^{(3)}R$ is the curvature scalar appropriate to the initial $t = \text{constant}$ hypersurface $\mathscr{S}$ and K^2 is the determinant of the metric g_{ab} on $\mathscr{S}$. The bar denotes covariant differentiation with respect to the metric g_{ab}. The momenta p^{ab} canonically conjugate to the g_{ab} are given by

$$p^{ab} \equiv \tfrac{1}{2}K(v^{ab} - e^{ab}v)$$

where

$$e^{ab} \equiv K^{-2}\,\text{cofactor}\,(g_{ab})$$

$$v_{ab} \equiv 2\Gamma^0_{ab}(-g^{00})^{-1/2} = \frac{dg_{ab}}{d\tau_L} = \text{SECOND FUNDAMENTAL FORM ON } \mathscr{S}$$

$$v \equiv g_{ab}v^{ab}$$

$d\tau_L$ represents the infinitesimal distance in the direction of the unit normal $l_\mu(l_\mu l^\mu = -1)$ to $\mathscr{S}$. The primary constraints generate changes in the $g_{0\mu}$, i.e., in the scaling and orientation (relative to l_μ) of the t axis. The secondary constraints generate changes in the co-ordinate system on $\mathscr{S}$:

$$\underset{\xi^\mu}{\mathscr{L}}\, \mathscr{O}(g_{ab}, p^{ab}) = [\mathscr{O}, H[\xi^\mu]] \tag{1.3}$$

where

$$H[\xi^\mu] \equiv \int d^3x\,\xi^\mu \mathscr{H}_\mu \equiv \int d^3x(\xi^L \mathscr{H}_L + \xi_r e^{rs}\mathscr{H}_s)$$

$$\mathscr{H}_\mu \equiv l_\mu \mathscr{H}^L + e_\mu{}^s \mathscr{H}_s = -l_\mu \mathscr{H}_L + e_\mu{}^s \mathscr{H}_s$$

$$e_\mu{}^s \equiv \delta^s_\mu + l_\mu l^s$$

and $\mathscr{O}$ is any function of the canonical variables g_{ab} and p^{ab}. The Hamiltonian is just that particular combination of the constraints which generates the time translations:

$$H \equiv H[\xi^\mu = \delta^\mu_0] = \int d^3x\{(-g^{00})^{-1/2}\mathscr{H}_L + g_{r0}e^{rs}\mathscr{H}_s\} \approx 0$$

In the Dirac formalism the four '0μ' degrees of freedom at each point of $\mathscr{S}$ have in effect been eliminated from the initial value problem. The $p^{0\mu}$, according to (1.1), must vanish, and their canonical conjugates g_{00} and g_{0r}, which appear in the Hamiltonian, are arbitrary functions which specify the purely conventional scaling and orientation of the time axis. Neither the $g_{0\mu}$ nor the $p^{0\mu}$ contain information about the real gravitational field, and so one can afford to ignore them. The remaining canonical

† In this paper Latin indices range from 1 to 3 and Greek indices range from 0 to 3; the signature of the metric is $-+++$.

variables g_{ab} and p^{ab} are invariant under the transformations generated by the primary constraints, as is evident from (1.1) and the standard canonical bracket relations. Thus in eliminating the '0μ' degrees of freedom, Dirac has also eliminated the primary constraints.

If, having eliminated the primary constraints in the manner proposed by Dirac, we could eliminate the secondary constraints as well, we would arrive at last at a complete set of canonical variables that are neither dependent nor redundant (the 'reduced phase space' variables). With such variables the original difficulties raised by the constraints could not appear. Attempts to carry out this program have taken two basic directions.

One technique is to set co-ordinate conditions. As Dirac has shown, by tying down the co-ordinate system it is possible, in theory at least, to eliminate the secondary constraints and with them all the remaining superfluous degrees of freedom at any given point of $\mathscr{S}$ (Dirac, 1959). The problem with this approach is that it is not possible to cover every Ricci-flat manifold with a single co-ordinate system. As a result, no matter what co-ordinate conditions one chooses, co-ordinate singularities arise and the technique ultimately breaks down.

The second approach is in effect a generalization of Dirac's method for eliminating the primary constraints. The idea in this case is to solve the secondary constraint equations for four of the momenta at each space point. Once a solution has been obtained one may eliminate the final four co-ordinate dependent degrees of freedom at each space point from the formalism by a procedure analogous to Dirac's elimination of the '0μ' degrees of freedom. As they stand, however, the secondary constraints (1.2) constitute a formidable set of coupled functional differential equations. It is neither desirable nor necessary to try to solve the constraint equations in this form. Since the constraints are the generators of infinitesimal co-ordinate transformations, the functional form of the constraints depends upon the transformation properties of the canonical variables under co-ordinate changes. The simpler the behavior of the canonical variables under changes in the co-ordinates on $\mathscr{S}$, the simpler the form of the constraint generators of these transformations will be. One can therefore hope to bring the constraint equations into a form much more amenable to solution by transforming to a more appropriate set of canonical variables. It is this strategy that we shall examine below.

2. *Simplifying the Secondary Constraints by Canonical Transformations*

Obviously it is possible to arrive at a wide variety of forms for the constraints by means of canonical transformations. There is, however, a limit to what canonical transformations can accomplish. Canonical transformations must preserve the Poisson bracket relations between the constraints.†

† This may be seen most simply from the fact that the Poisson bracket of two constraint generators of infinitesimal co-ordinate transformations is the generator of the commutator transformation, no matter which set of canonical variables we choose.

For the constraints (1.2) that we have to deal with these bracket relations are (DeWitt, 1967)

$$\left[\int d^3x'\,\xi^{r'}\mathscr{H}'_r,\ \int d^3x''\,\eta^{s''}\mathscr{H}''_s\right] = \int d^3x(\eta^r{}_{,s}\xi^s - \xi^r{}_{,s}\eta^s)\mathscr{H}_r \tag{2.1a}$$

$$\left[\int d^3x'\,\xi^{s'}\mathscr{H}'_s,\ \int d^3x''\,\eta^{L''}\mathscr{H}''_L\right] = \int d^3x\,\eta^L{}_{,s}\xi^s\mathscr{H}_L \tag{2.1b}$$

$$\left[\int d^3x'\,\xi^{L'}\mathscr{H}'_L,\ \int d^3x''\,\eta^{L''}\mathscr{H}''_L\right] = \int d^3x(\eta^L{}_{,s}\xi^L - \xi^L{}_{,s}\eta^L)\mathscr{H}^s \tag{2.1c}$$

In the work that follows we shall use the requirement that relations (2.1) be preserved under canonical transformations to make general statements about the degree of simplification of the secondary constraints that is possible.

According to (2.1), all the Poisson brackets between secondary constraints are zero. But they do not vanish identically (strongly), as is the case for the primary constraints; rather they are proportional to constraints. This peculiar characteristic of the secondary constraints is an indication of their relative complexity. Although with Dirac's choice of canonical variables the primary constraints take the simple form (1.1) of canonical momenta, there exists no set of canonical variables for which this can be true of the secondary constraints. For the Poisson brackets between canonical momenta vanish strongly, while the Poisson brackets between the secondary constraints do not. Thus we cannot eliminate the secondary constraints by a simple repetition of Dirac's procedure for eliminating the primary constraints.

If we cannot write the secondary constraint equations in the trivial form of (1.1), it is natural to ask whether it is at least possible to write them in some form in which they can be solved by purely algebraic means, i.e., in which no derivatives or integrals of the canonical momenta appear. To answer this question we must check whether there exist any such forms consistent with (2.1). In the work that follows we shall confine ourselves to canonical transformations which preserve the configuration space subspace of Dirac's original phase space. While other canonical transformations might conceivably be useful in simplifying the constraints, an analysis of the more general problem from the above point of view presents serious difficulties.

Consider first the spatial constraints (1.2a). $\mathscr{H}_s$, as it stands, is a linear function of the momenta. Under canonical transformations for which the new co-ordinates q^A $(A = 1,\ldots,6)$ are functions of the g_{ab} only, this linearity is preserved, for the new momenta p_A must be linear functions of the p^{ab}. We therefore seek permissible forms for $\mathscr{H}_s$ which are linear as well as algebraic in the p_A:

$$\mathscr{H}_s = F^A{}_s p_A + G_s \approx 0 \tag{2.2}$$

Here $F^A{}_s$ and G_s are functions of the new canonical co-ordinates q^A only. A short calculation shows that (2.1a) is true for (2.2) if and only if

$$F^A{}_s = q^A{}_{,s}$$

$$G_s = q^A{}_{,s} G_{,A}$$

where G is any algebraic function of the q^A and

$$G_{,A} \equiv \frac{\partial G}{\partial q^A}$$

Thus if $\mathscr{H}_s$ is to be algebraic function of the canonical momenta, it must have the form

$$\mathscr{H}_s = q^A{}_{,s}(p_A + G_{,A}) \approx 0 \quad (A = 1, \ldots 6) \tag{2.3}$$

With $\mathscr{H}_s$ in this form, relations (1.3) imply

$$\underset{\xi^s}{\mathscr{L}}\, q^A = \left[q^A, \int d^3x'\, \xi^{s'} \mathscr{H}'_s\right] = q^A{}_{,s}\xi^s$$

$$\underset{\xi^s}{\mathscr{L}}\, p_A = \left[p_A, \int d^3x'\, \xi^{s'} \mathscr{H}'_s\right] = (p_A \xi^s)_{,s} + (\xi^s G_{,A})_{,s} - q^B{}_{,s} G_{,AB}\, \xi^s$$

In other words, the new canonical co-ordinates q^A must behave as 3-scalars with respect to co-ordinate transformations within $\mathscr{S}$, while the transformation properties of the conjugate momenta p_A depend on the particular form of G. If, for instance, $G = 0$, the p_A transform as 3-scalar densities of weight +1.

Given the above hint it is a simple matter to find an explicit canonical transformation which brings $\mathscr{H}_s$ into the form (2.3). In fact any transformation that introduces 3-scalars as canonical co-ordinates will achieve this. If, for instance, we choose as new canonical co-ordinates an arbitrary set of independent 3-scalar functions of the g_{rs},

$$q^A = q^A(g_{rs}) \tag{2.4a}$$

and as new canonical momenta the conjugate 3-scalar densities,

$$p_A = \int [g'_{ab}, p_A]\, p^{ab'}\, d^3x' \Leftrightarrow p^{ab} = \int [q^{A'}, p^{ab}]\, p'_A\, d^3x' \tag{2.4b}$$

then we have by explicit substitution in $\mathscr{H}_s$

$$\mathscr{H}_s = -2 g_{sa} p^{ab}|_b$$

$$\mathscr{H}_s = -2 \int [q^{A'}, g_{sa} p^{ab}|_b]\, p'_A d^3x'$$

$$\mathscr{H}_s = \int [q^{A'}, \mathscr{H}_s]\, p'_A\, d^3x'$$

But since the q^A are 3-scalars,

$$\left[q^A, \int d^3x'\, \xi^{s'} \mathscr{H}'_s\right] = \mathscr{L}_{\xi^s} q^A = q^A{}_{,s}\xi^s \Rightarrow [q^{A'}, \mathscr{H}_s] = q^A{}_{,s}\delta(x - x')$$

Thus we finally obtain

$$\mathscr{H}_s = q^A{}_{,s} p_A \approx 0$$

In this case $G = 0$. We get the more general expression for $\mathscr{H}_s$ by replacing p_A with $\tilde{p}_A = p_A - G_{,A}$.

As far as conditions (2.1a) are concerned, the range of the index A is irrelevant. Peres has shown that it is possible to choose as many as three of the canonical co-ordinates and momenta at each space point to be 3-invariants (Peres, 1968). If the remaining canonical co-ordinates are 3-scalars, $\mathscr{H}_s$ will have the form (2.3) but with $A = 1, 2$, and 3 only; the 3-invariant canonical variables cannot appear in $\mathscr{H}_s$.

With the spatial constraint equations in the simple form (2.3), it appears to be a trivial matter to solve them. However, a certain amount of care is necessary here. One can solve for three of the p_A, say p_a, only at those points of $\mathscr{S}$ where the Jacobian determinant $|q^a{}_{,s}|$ does not vanish:

$$|q^a{}_{,s}| \neq 0 \tag{2.5}$$

If (2.5) holds at any particular point of $\mathscr{S}$, the 3-scalar q^a are valid intrinsic co-ordinates on $\mathscr{S}$ at that point. Thus in order to solve (2.3) we must in effect construct a valid set of intrinsic spatial co-ordinates at every point of $\mathscr{S}$.

Next we turn to the remaining constraints (1.2b). $\mathscr{H}_L$ is originally a quadratic function of the momenta, and, by the same argument as before, this property is preserved under the class of canonical transformations we are considering. Consequently permissible expressions for $\mathscr{H}_L$ which are algebraic functions of the momenta must have the general form

$$\mathscr{H}_L = F^{AB} p_A p_B + F^A p_A + F \approx 0 \tag{2.6}$$

where F, F^A, and F^{AB} are functions of the q^A only. But, with $\mathscr{H}_s$ given by (2.3), there exists no expression of the general form (2.6) for $\mathscr{H}_L$ which satisfies (2.1). We can see this from the following argument. Substitution of (2.3) and (2.6) into (2.1c) leads to the requirements that F^A and F^{AB} be purely algebraic functions of the q^A and that F satisfy

$$[p'_A, F''] = -\tfrac{1}{2} F'_{AB} q^{B'}{}_{,s'} e^{rs'}(q)\, \delta_{,r''}(x' - x'') + \sigma'_A(q)\, \delta(x' - x'') \tag{2.7}$$

where σ_A is some unspecified function of the q^A and F_{AB} is defined by

$$F_{AB} F^{BC} = \delta^C_A$$

But no F satisfying the above relations exists, for (2.7) is inconsistent with the Jacobi identity

$$[p_A, [p'_B, F'']] + [p'_B, [F'', p_A]] + [F'', [p_A, p'_B]] = 0$$

We conclude that it is not possible to bring $\mathscr{H}_L$ into the form (2.6) if $\mathscr{H}_s$ has the form (2.3). Any transformation which preserves the original configuration space and reduces the spatial constraints to the form (2.3) will be so complicated as to destroy the simple algebraic character of the momentum dependence of $\mathscr{H}_L$ in (1.2b). In the case of transformations of the type (2.4), for instance, the nonlocal form of (2.4b) implies that the algebraic momentum dependence in (1.2b) is not preserved.

3. *Summary and Conclusion*

In the above work we have obtained two negative results. First of all, there exists no canonical transformation whatever which will reduce all the constraints to the trivial form of canonical momenta. And secondly, no canonical transformation which preserves the Dirac configuration space will reduce all the constraints to purely algebraic functions of the momenta. These results, which follow directly from the Poisson bracket relations (2.1), indicate that the problem of solving the constraint equations is a difficult one. It appears likely that no very profound simplification of all the constraints is possible; probably it is necessary to solve some kind of functional differential equation at each point of $\mathscr{S}$ in order to eliminate all the redundant degrees of freedom from the Dirac formalism.

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